21 Telegrapher's equation

> Information is power, and those that have access to it are powerful.
>
> Senator Fred Thompson

> In vain Whitehouse used his two thousand volt induction coils to try to push messages through faster — after four weeks of this treatment the cable gave up the ghost; 2500 tons of cable and £350000 of capital lay useless on the ocean floor.
>
> T. W. Körner, *Fourier Analysis.*

A lot of exciting details can be found in the book by Körner, I here give only brief synopsis of the story.

The electric telegraph was invented in 1830, and immediately cable started connecting the main cities of Europe. In 1850 a cable connected London and France (after 12 hours of functioning the cable was accidentally cut by a ship), for the second attempt a much heavier cable was used, since it was discovered that the signal through under water cable cannot be transmitted as fast as along the cable on the ground. Faraday predicted this effect because of increased capacitance of undersea cables. Eventually it was understood that it is important to have a cable connecting USA and England — 2500 miles long cable was required to make this possible. The first attempt was made in 1857, the cable was snipped after 335 miles. In 1858 a new attempt to lay the cable failed because of the storm. Finally, in 1858 a cable was laid, and on August 16th it took 16 and a half hours to receive 90 words greeting (stocks of the company that owned the cable went down); what happened next with the cable is described in the quotation. In 1864 another attempt was made, Western Union company decided to put the cable through Russia. At the same time, in 1965 another company tried to do it again through the Atlantic ocean, and after 1250 miles, the cable parted with the ship. Finally, in 1866 new cable connected New-York and London. Due to some mechanical adjustments this time it was actually possible to send and receive signals.

Now I will try to explain what happened with the signal initially, and how engineers actually solved the problem.

Let me introduce the following variables: i is the current, v is the potential, L is the inductance, C is the capacitance, R is the resistance, G is the leakage conductance. If I apply the usual physical laws that describe the change of the current and potential in my cable, I end up with the system of first order partial differential equations:

$$
\begin{aligned}
i_x + Cv_t + Gv = 0,\\
v_x + Li_t + Ri = 0,
\end{aligned}
$$

which is called the system of telegrapher's equations. I assume that I have no boundary conditions: $-\infty < x < \infty$. If I differential the first equation with respect to t and the second one with respect to x, I can write that

$$
v_{xx} + Li_{tx} + Ri_x = R(-Cv_t - Gv) + v_{xx} + L(-Cv_{tt} - Gv_t) = 0,
$$

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or, after rearranging

$$(RC + LG)v_t + LCv_{tt} + GRv = v_{xx},$$

which is now a linear second order PDE, which resembles both the heat and the wave equations, which we know how to solve! To analyze this equation, I first would like to get rid of one term. That is, I introduce new variable

$$v = ue^{-\alpha t},$$

where α will be determined later. I find

$$\begin{aligned}
v_{xx} &= u_{xx}e^{-\alpha t}, \\
v_t &= u_t e^{-\alpha t} - \alpha u e^{-\alpha t}, \\
v_{tt} &= u_{tt}e^{-\alpha t} - 2\alpha u_t e^{-\alpha t} + \alpha^2 u e^{-\alpha t}.
\end{aligned}$$

Plugging this expressions into my equation (and canceling exponents) I get

$$\begin{aligned}
u_{xx} &= (RC + LG)(u_t - \alpha u) + RGu + LC(u_{tt} - 2\alpha u_t + \alpha^2 u) = \\
&= LCu_{tt} + u_t(RC + LG - LC2\alpha) + u(-\alpha(RC + LG) + \alpha^2 LC).
\end{aligned}$$

That is, if I choose my α as

$$\alpha = \frac{RC + LG}{2LC} = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right),$$

then my equation becomes

$$u_{tt} = a^2 u_{xx} + b^2 u,$$

where

$$a^2 = \frac{1}{LC}, \quad b^2 = \frac{1}{4}\left(\frac{G}{C} - \frac{R}{L}\right)^2.$$

To understand what is happening with my signal, I will look for a solution to my telegrapher's equation in the form of a traveling wave:

$$u(t, x) = f(x - ct) = f(\xi).$$

I know that the wave equation has perfectly nice traveling wave solutions, and this is how I would like my signal to propagate. After plugging this ansatz I have

$$(a^2 - c^2)f'' + b^2 f = 0,$$

where the derivative now is taken with respect to variable ξ.

I have several possible cases. First, $b = 0$ and $c = \pm a$ implies that *any* f can be a traveling wave solution, and therefore one can pass any signal with the fixed speed a. In case of $b > 0$ I have the characteristic polynomial

$$(a^2 - c^2)\lambda^2 + b^2 = 0.$$

If $(a^2 - c^2) < 0$ we have unbounded solutions — physically not realistic. Hence I assume that $a > |c|$, which would give me

$$\lambda_{1,2} = \pm \frac{b}{\sqrt{a^2 - c^2}}\mathrm{i}.$$

This implies that my only allowable solutions are

$$f(\xi) = \cos k\xi, \quad f(\xi) = \sin k\xi,$$

where

$$k = \frac{b}{\sqrt{a^2 - c^2}}\,,$$

which is called *the wave number*. The wave length (this is the same as wave period if my wave depends on the time variable) is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{b}\sqrt{a^2 - c^2}\,,$$

which means that in this case the wave length must satisfy (and hence not every wave length can be transmitted!)

$$0 < \lambda < \frac{2\pi a}{b}\,,$$

and each wave with a fixed length λ must travel with its own velocity

$$c_\lambda = \pm\sqrt{a^2 - \frac{b^2}{4\pi}\lambda^2}\,.$$

This implies that in the cable with $b \neq 0$ the signal must first be represented as a sum of several harmonics (remember, only sine and cosine are traveling wave solutions), which can always be done thanks to the Fourier series, but different harmonics move with different velocities and thus it is almost impossible to understand what was initially sent. Hence, to fight this effect, one can choose the parameters such that $b = 0$:

$$\frac{G}{C} = \frac{R}{L}\,,$$

and this will allow the signal to travel as a whole. Problem solved!